Homework Issues:

Accumulation points of $\left\{2^n + \frac{1}{k} : n = 1, 2, \cdots, k = 1, 2, \cdots\right\}$

1. ∞ is just symbol for convenience. In our class, it is not recognized as a number.

$\lim_{n\to\infty} a_n = +\infty$ means $\forall M > 0,\ \exists N \in \mathbb{Z}_+,\ \forall n > N,\ |a_n| > M.$

$\lim_{n\to\infty} a_n = \underset{\text{real number}}{a}$ means $\forall \varepsilon > 0,\ \exists N \in \mathbb{Z}_+,\ \forall n > N,\ |a_n - a| < \varepsilon$

DON'T YOU EVER WRITE ANYTHING "IS" or "ATTAINS" ∞

"goes to" or "approaches to" is legal.

2. $\left\{2^n + \frac{1}{k} : n = 1, 2, \cdots, k = 1, 2, \cdots\right\}$

$$= \bigcup_{n=1}^{\infty} \left\{2^n + \frac{1}{k} : k = 1, 2, \cdots\right\}.$$

Easy to see $\left\{2^n + \frac{1}{k} : k = 1, 2, \cdots\right\}$ has a unique accumulation point: 2^n.

$\Rightarrow \left\{2^n + \frac{1}{k} : n = 1, 2, \cdots, k = 1, 2, \cdots\right\}$ has accumulation points 2^n, $n = 1, 2, \cdots$.

Cut-off Trick:

Example: $\{a_n\}$ converges $\Rightarrow$ $\{a_n\}$ is bounded

Pf: Say $a_n \to a$, then

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ |a_n - a| < \varepsilon.$$

Fix this ε, then

$\{a_{N+1}, a_{N+2}, \cdots\}$ is bounded by $a-\varepsilon$ and $a+\varepsilon$.

Also since $\{a_1, \cdots, a_N\}$ is a finite set,

$\{a_1, \cdots, a_N\}$ is bounded by $P = \min_{1\le n\le N} a_n$, $Q = \max_{1\le n\le N} a_n$.

Therefore $\{a_1, a_2, \cdots, a_N, a_{N+1}, \cdots\}$ is bounded

by $\min\{P, a-\varepsilon\}$ and $\max\{Q, a+\varepsilon\}$

Hence $\{a_n\}$ is bounded.

Key of cut-off trick: Use the def of limit to cut the tail of inf. many terms off. That inf. tail will be well controlled. What remains is to control the the head consisting of finitely many terms.

1.31. Given $\{a_n\}_{n=1}^{\infty}$, set $\alpha_n = \dfrac{a_1+a_2+\cdots+a_n}{n}$ for every n.

If $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty} \alpha_n$ exists and is precisely A.

Given an example of $\{a_n\}_{n=1}^{\infty}$, s.t. $\{\alpha_n\}_{n=1}^{\infty}$ converges, but $\{a_n\}_{n=1}^{\infty}$ doesn't

$$\alpha_n - A = \frac{a_1+a_2+\cdots+a_n}{n} - \frac{A+A+\cdots+A}{n} \qquad A = \frac{nA}{n}$$

$$= \frac{1}{n}\Big((a_1-A)+(a_2-A)+\cdots+(a_n-A)\Big)$$

Trick: Cut-off.

For any $\varepsilon>0$, from $\lim_{n\to\infty} a_n = A$. $\exists N\in\mathbb{Z}_+, \forall n>N, |a_n - A| < \frac{\varepsilon}{2}$.

So $|a_{N+1}-A|+|a_{N+2}-A|+\cdots+|a_{N+m}-A| < \frac{m\varepsilon}{2}$. for every $m \geq 1$.

CONTROL THE TAIL $\Rightarrow$ $\frac{|a_{N+1}-A|+|a_{N+2}-A|+\cdots+|a_{N+m}-A|}{N+m} < \frac{m\varepsilon}{2(N+m)}$ $(*)$

$< \frac{\varepsilon}{2}$ (b/c $\frac{m}{N+m} < 1$)

Note that this fact works FOR ALL $m \geq 1$.

by letting m large. one can make.

CONTROL THE HEAD $\frac{|a_1-A|+|a_2-A|+\cdots+|a_N-A|}{N+m} < \frac{\varepsilon}{2}$

Rigorously, since $|a_1-A|+|a_2-A|+\cdots+|a_N-A|$ is a finite sum,

it's a number (depending on N).

Pick M to be an integer longer than $\frac{2(|a_1-A|+\cdots+|a_N-A|)}{\varepsilon}-N$.

So $\forall m > M$,

$$\frac{|a_1-A|+|a_2-A|+\cdots+|a_N-A|}{N+m} < \frac{\varepsilon}{2} \qquad (**)$$

i.e. For any $\varepsilon > 0$, pick $N' = N+M$, then $\forall\, n > N'$

$$|\alpha_n - A| = \left|\frac{a_1-A+a_2-A+\cdots+a_N-A}{n} + \frac{a_{N+1}-A+a_{N+2}-A+\cdots+a_n-A}{n}\right|$$

$$\leq \frac{1}{n}\left(|a_1-A|+\cdots+|a_N-A|\right) + \frac{1}{n}\left(|a_{N+1}-A|+\cdots+|a_n-A|\right)$$

from $(*)$ and $(**)$. $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Counter Example : $a_n = (-1)^n$. $\alpha_n = \frac{(-1)^1+(-1)^2+\cdots+(-1)^n}{n} = \begin{cases} 0 & n \text{ is even} \\ \frac{-1}{n} & n \text{ is odd.} \end{cases}$

$\lim_{n\to\infty} \alpha_n = 0$. $\lim_{n\to\infty} a_n$ DNE (does not exist)

For doubly-indexed sequences, in general it's not true that

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{mn} = \lim_{n\to\infty}\lim_{m\to\infty} a_{mn}.$$

Example:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & \cdots \\ \hline 1 & 1 & 2 & 3 & \cdots \\ 2 & 0 & 1 & 2 & \cdots \\ 3 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array} \qquad a_{mn} = \begin{cases} m+n-1 & m \leq n \\ 0 & m > n \end{cases}$$

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{mn} = 0, \quad \lim_{n\to\infty}\lim_{m\to\infty} a_{mn} = +\infty.$$

39. $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} y_n = x_0$. Define z_n: $z_{2n} = x_n$; $z_{2n-1} = y_n$.

Prove $\lim_{n\to\infty} z_n = x_0$.

$$|z_n - x_0| = \begin{cases} |x_k - x_0| & \text{if } n = 2k \\ |y_k - x_0| & \text{if } n = 2k-1 \end{cases}$$

For $\varepsilon > 0$, pick $K_1 > 0$, s.t. $\forall k > K_1$, $|x_k - x_0| < \varepsilon$.

pick $K_2 > 0$, s.t. $\forall k > K_2$, $|y_k - x_0| < \varepsilon$.

Therefore let $N = \max\{2K_1, 2K_2 - 1\}$

$$\forall n > N, \quad |z_n - x_0| = \begin{cases} |x_k - x_0| & n = 2k \\ |y_k - x_0| & n = 2k-1. \end{cases}$$

is strictly less than ε.

$$\left(\begin{array}{l} \text{if } n = 2k > N \geq 2K_1 \Rightarrow k > K_1 \Rightarrow |x_k - x_0| < \varepsilon \\ \text{if } n = 2k-1 > N \geq 2K_2 - 1 \Rightarrow k > K_2 \Rightarrow |y_k - x_0| < \varepsilon \end{array} \right).$$